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A Note on Interpolating Periodic Quintic Splines with Equally Spaced Nodes[1]

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1. INTRODUCTION

We begin with the definition of an interpolating quintic spline function, state the purpose of this communication and give a few appropriate references. Throughout the paper f is assumed to be a continuous and periodic function with period 1.

If nodes $0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$ are prescribed, then an interpolating periodic quintic spline ϕ, associated with f, is a function with the following properties:

(a) ϕ and f have the same values at the nodes, i.e. $\phi_i = f_i$ $(i = 1, 2, \ldots, n)$;
(b) ϕ is four times continuously differentiable and $\phi_0^{(i)} = \phi_n^{(i)}$, $(i = 1, 2, 3, 4)$;
(c) ϕ reduces to a polynomial of degree at most five on each subinterval $[x_{i-1}, x_i]$.

Under the assumption that the points x_i are equally spaced on $[0, 1]$, this note gives some estimates for the magnitude of the various spline derivatives at the nodes and uses these results to establish two approximation theorems. Moreover, an upper bound for the norm of the quintic spline operator is derived.

This paper was motivated by the first sections of Chapter IV of the book by Ahlberg, Nilson and Walsh [*1*], where polynomial splines of arbitrary degree are treated. The articles [*2*], [*3*] and [*4*] are also related because similar problems for the cubic splines are dealt with there. Moreover, [*4*] contains a discussion of deficient quintic splines.

2. A PROCEDURE FOR CONSTRUCTING QUINTIC SPLINES

In this section we show how quintic splines can be constructed using Hermite interpolation. The restriction that the nodes are equally spaced is not necessary yet.

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Define $\phi_i(x) \equiv \phi(x)$ for $x_{i-1} \leqslant x \leqslant x_i$.

If we introduce the notation

$$h_i = x_i - x_{i-1}, \qquad \lambda_i = \phi_i', \qquad \mu_i = \phi_i'', \qquad m_i = \phi_i^{(3)}, \qquad M_i = \phi_i^{(4)}, \quad (1)$$

then on the interval $[x_{i-1}, x_i]$ the quintic spline can be written in the following form

$$\phi_i(x) = f_{i-1} A_i(x) + f_i B_i(x) + \lambda_{i-1} C_i(x) + \lambda_i D_i(x) + \mu_{i-1} E_i(x) + \mu_i F_i(x). \quad (2)$$

Here $A_i(x), \ldots, F_i(x)$ are quintic polynomials determined by the following identity:

For an arbitrary quintic polynomial $P(x)$ we have

$$P(x) \equiv P_{i-1} A_i(x) + P_i B_i(x) + P_{i-1}' C_i(x) + P_i' D_i(x) + P_{i-1}'' E_i(x) + P_i'' F_i(x) \quad (3)$$

where we set

$$P^{(k)}(x_i) = P_i^{(k)}, \qquad P^{(k)}(x_{i-1}) = P_{i-1}^{(k)}, \qquad k = 0, 1, 2.$$

If $A(t), \ldots, F(t)$ denote the polynomials of (3) when $[x_{i-1}, x_i]$ is replaced by $[0, 1]$, we have

$$\begin{aligned} A(t) &= (1-t)^3 (6t^2 + 3t + 1), \qquad & B(t) &= A(1-t) \\ C(t) &= t(1-t)^3 (1+3t), & D(t) &= -C(1-t) \\ E(t) &= \tfrac{1}{2} t^2 (1-t)^3, & F(t) &= E(1-t). \end{aligned}$$

The expressions for $A_i(x), \ldots, F_i(x)$ are now obtained by setting $t = (x - x_{i-1}) h_i^{-1}$, multiplying $C_i(x)$ and $D_i(x)$ by h_i and $E_i(x)$ and $F_i(x)$ by h_i^2.

We have $A_i(x)$, $B_i(x)$, $C_i(x)$, $E_i(x)$, $F_i(x) \geqslant 0$, whereas $D_i(x) \leqslant 0$ on $[x_{i-1}, x_i]$. Moreover,

$$A_i(x) + B_i(x) = 1. \quad (4)$$

If the nodes are equally spaced, i.e. $h_i = (1/n)\,(i = 1, 2, \ldots, n)$, then elementary calculations show that

$$\begin{aligned} C_i(x) - D_i(x) &= (x - x_{i-1})(1 - n(x - x_{i-1})) \cdot \\ &\quad \cdot \{1 + n(x - x_{i-1}) - n^2 (x - x_{i-1})^2\} \leqslant \frac{5}{16n} \end{aligned} \quad (5)$$

and

$$E_i(x) + F_i(x) = \tfrac{1}{2}(x - x_{i-1})^2 (1 - n(x - x_{i-1}))^2 \leqslant \frac{1}{32n^2}. \quad (6)$$

These two results will be needed later on.

Formula (2), together with (3), assures that the function ϕ is twice continuously differentiable on [0,1]. In order that condition (b) will be completely satisfied, it is necessary that we have

$$\lim_{x \uparrow x_i} \phi_i^{(3)}(x) = \lim_{x \downarrow x_i} \phi_{i+1}^{(3)}(x)$$

and

$$\lim_{x \uparrow x_i} \phi_i^{(4)}(x) = \lim_{x \downarrow x_i} \phi_{i+1}^{(4)}(x).$$

This is identical with

$$f_{i-1} A_i^{(k)}(x_i) + \ldots + \mu_i F_i^{(k)}(x_i) = f_i A_{i+1}^{(k)}(x_i) + \ldots + \mu_{i+1} F_{i+1}^{(k)}(x_i), \qquad (k = 3, 4). \tag{7}$$

It is thus necessary to calculate the third and fourth derivatives of the polynomials $A_i(x) \ldots F_i(x)$. This is rather tedious, yet straightforward. We obtain (cf. [*1*], p. 120)

$$\left.\begin{aligned}
A_i^{(3)}(x_i) &= -60h_i^{-3}, & A_{i+1}^{(3)}(x_i) &= -60h_{i+1}^{-3},\\
A_i^{(4)}(x_i) &= -360h_i^{-4}, & A_{i+1}^{(4)}(x_i) &= 360h_{i+1}^{-4},\\
B_i^{(3)}(x_i) &= 60h_i^{-3}, & B_{i+1}^{(3)}(x_i) &= 60h_{i+1}^{-3},\\
B_i^{(4)}(x_i) &= 360h_i^{-4}, & B_{i+1}^{(4)}(x_i) &= -360h_{i+1}^{-4},\\
C_i^{(3)}(x_i) &= -24h_i^{-2}, & C_{i+1}^{(3)}(x_i) &= -36h_{i+1}^{-2},\\
C_i^{(4)}(x_i) &= -168h_i^{-3}, & C_{i+1}^{(4)}(x_i) &= 192h_{i+1}^{-3},\\
D_i^{(3)}(x_i) &= -36h_i^{-2}, & D_{i+1}^{(3)}(x_i) &= -24h_{i+1}^{-2},\\
D_i^{(4)}(x_i) &= -192h_i^{-3}, & D_{i+1}^{(4)}(x_i) &= 168h_{i+1}^{-3},\\
E_i^{(3)}(x_i) &= -3h_i^{-1}, & E_{i+1}^{(3)}(x_i) &= -9h_{i+1}^{-1},\\
E_i^{(4)}(x_i) &= -24h_i^{-2}, & E_{i+1}^{(4)}(x_i) &= 36h_{i+1}^{-2},\\
F_i^{(3)}(x_i) &= 9h_i^{-1}, & F_{i+1}^{(3)}(x_i) &= 3h_{i+1}^{-1},\\
F_i^{(4)}(x_i) &= 36h_i^{-2}, & F_{i+1}^{(4)}(x_i) &= -24h_{i+1}^{-2}.
\end{aligned}\right\} \tag{8}$$

If we insert these values in formula (7), we arrive at the following two basic systems of equations ($i = 1,2,\ldots,n$):

$$\begin{aligned}
&8\lambda_{i-1} h_i^{-2} - 12\lambda_i(h_{i+1}^{-2} - h_i^{-2}) - 8\lambda_{i+1} h_{i+1}^{-2} + \mu_{i-1} h_i^{-1} + \\
&\quad -3\mu_i(h_{i+1}^{-1} + h_i^{-1}) + \mu_{i+1} h_{i+1}^{-1} = 20\{(f_i - f_{i-1}) h_i^{-3} - (f_{i+1} - f_i) h_{i+1}^{-3}\},
\end{aligned} \tag{9}$$

$$\begin{aligned}
&14\lambda_{i-1} h_i^{-3} + 16\lambda_i(h_{i+1}^{-3} + h_i^{-3}) + 14\lambda_{i+1} h_{i+1}^{-3} + 2\mu_{i-1} h_i^{-2} + \\
&\quad + 3\mu_i(h_{i+1}^{-2} - h_i^{-2}) - 2\mu_{i+1} h_{i+1}^{-2} = 30\{(f_{i+1} - f_i) h_{i+1}^{-4} + (f_i - f_{i-1}) h_i^{-4}\}.
\end{aligned} \tag{10}$$

3. Some Basic Relations

From now on we assume that the nodes are uniformly distributed on $[0, 1]$, i.e. $h_i = 1/n$ for all i. Equations (9) and (10) are simplified and take the following form:

$$8n\lambda_{i-1} - 8n\lambda_{i+1} + \mu_{i-1} - 6\mu_i + \mu_{i+1} = 20n^2(-f_{i-1} + 2f_i - f_{i+1}),$$
$$7n\lambda_{i-1} + 16n\lambda_i + 7n\lambda_{i+1} + \mu_{i-1} - \mu_{i+1} = 15n^2(f_{i+1} - f_{i-1}).$$

If we write down these relations, respectively, for the points x_{i-1}, x_i and x_{i+1}, we can eliminate the parameters μ_{i-2}, μ_{i-1}, μ_i, μ_{i+1} and μ_{i+2} and obtain an equation in which only the first derivatives of the spline are involved. Moreover, the same can be done by eliminating the parameters λ_i.

Under the assumption that all indices which occur are interpreted modulo n, we get

$$\lambda_{i-2} + 26\lambda_{i-1} + 66\lambda_i + 26\lambda_{i+1} + \lambda_{i+2} = 5n(f_{i+2} + 10f_{i+1} - 10f_{i-1} - f_{i-2}) \tag{11}$$

and also

$$\mu_{i-2} + 26\mu_{i-1} + 66\mu_i + 26\mu_{i+1} + \mu_{i+2} = 20n^2(f_{i+2} + 2f_{i+1} - 6f_i + 2f_{i-1} + f_{i-2}). \tag{12}$$

There is one such equation for each value of $i = 1, 2, \ldots, n$. Because the matrix associated with these two systems of equations is diagonally dominant, there is a unique solution for the parameters λ_i and μ_i. Together with (2), this means that the interpolating quintic spline exists and is uniquely determined. We remark that this can also be deduced from a general existence theorem concerning polynomial splines of odd degree ([*1*], p. 135).

Relations similar to (11) and (12) are valid for the third and fourth derivatives m_i and M_i. Using (2) and (8), we note that

$$m_{i-1} = -60n^3 f_{i-1} + 60n^3 f_i - 36n^2\lambda_{i-1} - 24n^2\lambda_i - 9n\mu_{i-1} + 3n\mu_i,$$
$$m_i = -60n^3 f_{i-1} + 60n^3 f_i - 24n^2\lambda_{i-1} - 36n^2\lambda_i - 3n\mu_{i-1} + 9n\mu_i.$$

It is useful to draw a few consequences of the above formulae. If we express the parameters μ_i in terms of the λ_i and the m_i, and make use of the fact that the second spline derivative is continuous at the nodes, we obtain

$$\mu_i = \tfrac{5}{2}n^2(f_{i+1} - 2f_i + f_{i-1}) + \tfrac{3}{4}n(\lambda_{i-1} - \lambda_{i+1}) + \frac{1}{48n}(m_{i+1} - m_{i-1}), \tag{13}$$

$$3\lambda_{i+1} + 14\lambda_i + 3\lambda_{i-1} = 10n(f_{i+1} - f_{i-1}) + \frac{1}{12n^2}(m_{i-1} - 6m_i + m_{i+1}), \tag{14}$$

$$7\lambda_{i+1} + 26\lambda_i + 7\lambda_{i-1} = 20n(f_{i-1} - f_{i+1}) + \frac{1}{12n^2}(-3m_{i-1} + 2m_i - 3m_{i+1}).$$

If we eliminate the parameters λ_i from the last two equations, we arrive at

$$m_{i-2}+26m_{i-1}+66m_i+26m_{i+1}+m_{i+2}=60n^3(f_{i+2}-2f_{i+1}+2f_{i-1}-f_{i-2}). \tag{15}$$

Along the same lines there follows

$$M_{i-2}+26M_{i-1}+66M_i+26M_{i+1}+M_{i+2}= \\ =120n^4(f_{i+2}-4f_{i+1}+6f_i-4f_{i-1}+f_{i-2}); \tag{16}$$

this relation is contained in [*1*], p. 127.

Remark. Relations (12) and (16) may also be obtained by applying the interpolation method of Lidstone and carrying out an analysis similar to that of Section 2.

4. Statement and Proof of the Theorems

The first theorem of this section gives estimates of the magnitude for the various spline derivatives at the nodes.

For the sake of brevity, let

$$\omega_r\left(\frac{1}{n}\right)=\max_{|u-v|\leqslant 1/n}|f^{(r)}(u)-f^{(r)}(v)|,\quad \omega\left(\frac{1}{n}\right)=\omega_0\left(\frac{1}{n}\right),$$
$$a_i^{(r)}=\phi_i^{(r)}-f_i^{(r)}.$$

Theorem 1. *If the numbers λ_i, μ_i, m_i and M_i are defined as in* (1), *then*

$$\max_i|\lambda_i|\leqslant\frac{25}{6}n\ \omega\left(\frac{1}{n}\right), \tag{17}$$

$$\max_i|\mu_i|\leqslant\frac{145}{12}n^2\omega\left(\frac{1}{n}\right), \tag{18}$$

$$\max_i|m_i|\leqslant 20n^3\omega\left(\frac{1}{n}\right), \tag{19}$$

$$\max_i|M_i|\leqslant 160n^4\omega\left(\frac{1}{n}\right). \tag{20}$$

Proof. To establish (19), we assume that $\max_i|m_i|=|m_k|$. In view of (15), we can write

$$66|m_k|\leqslant 60n^3(|f_{k+2}-f_{k+1}|+|f_{k+1}-f_{k-1}|+|f_{k-1}-f_{k-2}|)+ \\ +|m_{k+2}|+26|m_{k+1}|+26|m_{k-1}|+|m_{k-2}|\leqslant \\ \leqslant 240n^3\omega\left(\frac{1}{n}\right)+54|m_k|.$$

Hence

$$12|m_k| \leqslant 240n^3\,\omega\left(\frac{1}{n}\right)$$

and (19) follows.

Using (16), the proof of (20) is similar. The inequality (17) is a consequence of (19) and (14); (18) follows from (19) and (13).

LEMMA. *If f belongs, respectively, to the classes C^4, C^3 and C^2 and has period* 1, *while ϕ is the interpolating quintic spline associated with f, then*

$$\max_i |a_i^{(4)}| \leqslant 25\omega_4\left(\frac{1}{n}\right), \tag{21}$$

$$\max_i |a_i^{(3)}| \leqslant 23\omega_3\left(\frac{1}{n}\right), \tag{22}$$

$$\max_i |a_i''| \leqslant 12\omega_2\left(\frac{1}{n}\right). \tag{23}$$

Proof. The above inequalities are proved by using relations (16), (15) and (12). We deduce only formula (21), because the method of proof is in all three cases similar.

Writing

$$n^4(f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2}) = f^{(4)}(\xi),$$

where

$$x_{i-2} < \xi < x_{i+2},$$

we obtain from (16), for $i = 1, 2, \ldots, n$,

$$\begin{aligned} a_{i+2}^{(4)} + 26a_{i+1}^{(4)} + 66a_i^{(4)} + 26a_{i-1}^{(4)} + a_{i-2}^{(4)} &= \\ = (f^{(4)}(\xi) - f_{i+2}^{(4)}) + 26(f^{(4)}(\xi) - f_{i+1}^{(4)}) &+ 66(f^{(4)}(\xi) - f_i^{(4)}) + \\ + 26(f^{(4)}(\xi) - f_{i-1}^{(4)}) &+ (f^{(4)}(\xi) - f_{i-2}^{(4)}). \end{aligned}$$

Assume $\max_i |a_i^{(4)}| = |a_k^{(4)}|$. We observe that the right-hand side of the above equation does not exceed $296\omega_4(1/n)$. Proceeding in a similar way as in the proof of Theorem 1, it follows that (21) holds.

THEOREM 2. *Let f, with period* 1, *belong to C^4, and let ϕ be the interpolating quintic spline of f. Then for all x,*

$$|\phi^{(4)}(x) - f^{(4)}(x)| \leqslant 26\omega_4\left(\frac{1}{n}\right), \tag{24}$$

$$|\phi^{(3)}(x) - f^{(3)}(x)| \leqslant 26n^{-1}\,\omega_4\left(\frac{1}{n}\right) + 23\omega_3\left(\frac{1}{n}\right), \tag{25}$$

$$|\phi''(x)-f''(x)| \leqslant 26n^{-2}\,\omega_4\left(\frac{1}{n}\right)+23n^{-1}\,\omega_3\left(\frac{1}{n}\right)+12\omega_2\left(\frac{1}{n}\right), \tag{26}$$

$$|\phi'(x)-f'(x)| \leqslant 26n^{-3}\,\omega_4\left(\frac{1}{n}\right)+23n^{-2}\,\omega_3\left(\frac{1}{n}\right)+12n^{-1}\,\omega_2\left(\frac{1}{n}\right), \tag{27}$$

$$|\phi(x)-f(x)| \leqslant 26n^{-4}\,\omega_4\left(\frac{1}{n}\right)+23n^{-3}\,\omega_3\left(\frac{1}{n}\right)+12n^{-2}\,\omega_2\left(\frac{1}{n}\right). \tag{28}$$

Proof. Formula (21) of the lemma implies that for all i,

$$|a_i^{(4)}| \leqslant 25\omega_4\left(\frac{1}{n}\right).$$

From this, and because of the fact that $\phi^{(4)}(x)$ is linear between the nodes, (24) follows. In order to prove (25), let $x_{i-1} \leqslant x < x_i$. We write

$$|\phi^{(3)}(x)-f^{(3)}(x)| = \left|\int_{x_{i-1}}^{x} (\phi^{(4)}(t)-f^{(4)}(t))\,dt + \phi_{i-1}^{(3)} - f_{i-1}^{(3)}\right|.$$

Relation (25) now follows immediately by applying the triangle inequality and then using (24) and (22). The proof of (26) makes use of formula (23) and is similar to the one we have just given.

Let again $x_{i-1} \leqslant x < x_i$. Since $\phi_{i-1} = f_{i-1}$ and $\phi_i = f_i$, there exists an η_i in (x_{i-1}, x_i), for which $f'(\eta_i) - \phi'(\eta_i) = 0$. Thus,

$$|\phi'(x)-f'(x)| = \left|\int_{\eta_i}^{x} (\phi''(t)-f''(t))\,dt\right|$$

and

$$|\phi(x)-f(x)| = \left|\int_{x_{i-1}}^{x} (\phi'(t)-f'(t))\,dt\right|.$$

In view of this, the inequalities (27) and (28) are a direct consequence of (26).

Remark. Following a similar procedure it is possible to replace the right-hand sides of the inequalities of Theorem 2 by expressions in which only $\omega_4(1/n)$ is involved.

THEOREM 3. *Let f, with period* 1, *belong to C, and let ϕ be the periodic interpolating quintic spline of f. Then for all x,*

$$|\phi(x)-f(x)| \leqslant \frac{27}{10}\,\omega\left(\frac{1}{n}\right).$$

Proof. Let $x \in [x_{i-1}, x_i]$. Using (2) and (4), we get

$$\phi_i(x)-f(x) = (f_{i-1}-f(x))\,A_i(x) + (f_i - f(x))\,B_i(x) + \\ + \lambda_{i-1}\,C_i(x) + \lambda_i\,D_i(x) + \mu_{i-1}\,E_i(x) + \mu_i\,F_i(x).$$

We observed already that all the functions $A_i(x)$, $B_i(x)$, ... are non-negative on the interval $[x_{i-1}, x_i]$, with the exception of $D_i(x)$. Therefore one has

$$|\phi_i(x) - f(x)| \leqslant \omega\left(\frac{1}{n}\right) + \max_j |\lambda_j| (C_i(x) - D_i(x)) + \\ + \max_j |\mu_j| (E_i(x) + F_i(x)).$$

Theorem 3 is now obtained by using inequalities (5), (6), (17) and (18).

Let $\phi = Lf$.

THEOREM 4. *An upper bound for the norm of the quintic spline operator L is given by*

$$\|L\| \leqslant \tfrac{22}{5}.$$

Proof. Select spline functions ϕ^i such that $\phi^i(x_j) = \delta_{ij}$. Then L has the form $Lf = \sum_{i=1}^n f_i \phi^i$. It can be shown that the norm of the operator L is now equal to the Chebyshev norm of the function $\sum_{i=1}^n |\phi^i|$. Select ξ such that $\|\sum_{i=1}^n |\phi^i|\| = \sum_{i=1}^n |\phi^i(\xi)|$, and let f be a continuous function of norm 1 which satisfies the equations $f(x_i) = \operatorname{sgn} \phi^i(\xi)$ and is linear in each interval $[x_{i-1}, x_i]$. Then $\|L\| = \|Lf\| = \|\phi\|$. From earlier analysis we have

$$\phi_i(x) = f_{i-1} A_i(x) + f_i B_i(x) + \lambda_{i-1} C_i(x) + \lambda_i D_i(x) + \mu_{i-1} E_i(x) + \mu_i F_i(x),$$

whence

$$|\phi_i(x)| \leqslant \|f\| (A_i(x) + B_i(x)) + \max_j |\lambda_j| (C_i(x) - D_i(x)) + \\ + \max_j |\mu_j| (E_i(x) + F_i(x)).$$

If we observe that $\omega(1/n) \leqslant 2$, and take into account the formulae (4), (5), (6), (17), (18), then Theorem 4 follows.

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